Closing Fri: 3.4(1),(2)
Closing Tues: 10.2
Closing next Fri: 3.5(1)(2)
Exams back on Tuesday

3.4 Chain Rule *(continued)*:

$$\frac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big)g'(x)$$

Also written as: $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

Entry Task: Find the derivatives of

1. $y = \sin^4(3x)$
2. $y = \sin(3x^4)$
3. $y = \tan(e^{2x} + \cos(x^3))$

$\boxed{1}$ $y = (\sin(3x))^4$

$$\boxed{\begin{aligned}\frac{dy}{dx} &= 4(\sin(3x))^3 \cdot \cos(3x) \cdot 3 \\ &= 12\cos(3x)\sin^3(3x)\end{aligned}}$$

$\boxed{2}$ $y = \sin(3x^4)$

$$\boxed{\begin{aligned}\frac{dy}{dx} &= \cos(3x^4) \cdot 12x^3 \\ &= 12x^3\cos(3x^4)\end{aligned}}$$

$\boxed{3}$ $y = \tan(e^{2x} + \cos(x^3))$

$$\boxed{\begin{aligned}\frac{dy}{dx} &= \sec^2(e^{2x} + \cos(x^3)) \cdot (e^{2x} \cdot 2 + -\sin(x^3)3x^2) \\ &= \sec^2(e^{2x} + \cos(x^3)) \cdot (2e^{2x} - 3x^2\sin(x^3))\end{aligned}}$$

Here is a brief "proof" of the chain rule:

$$
\begin{aligned}
\frac{d}{dx} f\big(g(x)\big) &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\
&= \lim_{h\to 0} \left(\frac{f(g(x+h)) - f(g(x))}{h} \, \frac{g(x+h) - g(x)}{g(x+h) - g(x)} \right) \\
&= \lim_{h\to 0} \left(\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \right) \left(\frac{g(x+h) - g(x)}{h} \right) \\
&= \underbrace{\lim_{h\to 0} \left(\frac{f\big(g(x+h)\big) - f\big(g(x)\big)}{g(x+h) - g(x)} \right)} \lim_{h\to 0} \left(\frac{g(x+h) - g(x)}{h} \right) \\
&= f'\big(g(x)\big) g'(x)
\end{aligned}
$$

NOTE: IF $r = g(x+h) - g(x)$

THEN $g(x+h) = g(x) + r$ AND AS $h \to 0$, $r \to 0$

$$\underbrace{\lim_{r\to 0} \frac{f(g(x)+r) - f(g(x))}{r}}_{f'(g(x))}$$

Identify the "first" rule you would use to differentiate these functions:
(sum, product, quotient or chain?)

1. $y = \sqrt{e^{4x} + x^2 + 1}$ CHAIN

2. $y = \dfrac{x^5}{\cos(5x+1)}$ QUOTIENT

3. $y = \sqrt[3]{x^3 + 1}\cos(\sin(5x))$ PRODUCT

4. $y = e^{\cot(x)} - 5(x^3 + 2)^{20}$ SUM

5. $y = \left(\dfrac{e^{2x}+1}{x^2+1}\right)^{10}$ CHAIN

$\boxed{1}$

$$y' = \frac{1}{2}\left(e^{4x} + x^2 + 1\right)^{-1/2} \cdot \left(4e^{4x} + 2x\right)$$

$\boxed{2}$

$$y' = \frac{\cos(5x+1)\,5x^4 - x^5(-\sin(5x+1))(5)}{\cos^2(5x+1)}$$

$$= \frac{5x^4(\cos(5x+1) + x\sin(5x+1))}{\cos^2(5x+1)}$$

$\boxed{3}$

$$y' = \sqrt[3]{x^3+1}\,(-\sin(\sin(5x)))\cos(5x)\cdot 5 + \frac{1}{3}(x^3+1)^{-2/3}\cdot 3x^2\cos(\sin(5x))$$

$\boxed{4}$

$$y' = e^{\cot(x)}(-\csc^2(x)) - 100(x^3+2)^{19}\cdot 3x^2$$

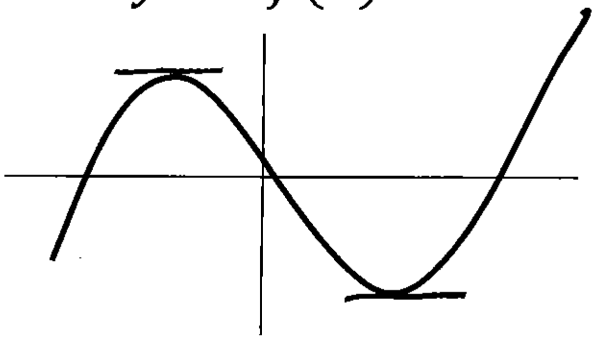
$$= -\csc^2(x)e^{\cot(x)} - 300x^2(x^3+2)^{19}$$

$\boxed{5}$

$$y' = 10\left(\frac{e^{2x}+1}{x^2+1}\right)^9 \cdot \frac{(x^2+1)2e^{2x} - (e^{2x}+1)2x}{(x^2+1)^2}$$

Standard Equations Calculus Review

Given $y = f(x)$



1. $\dfrac{dy}{dx} = f'(x) =$ slope of tangent.
2. Tangent line equation:

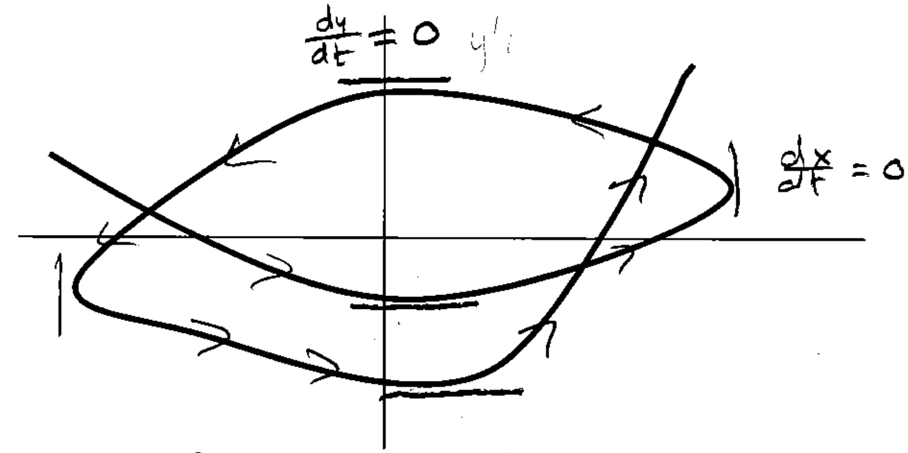
$$y = f'(a)(x - a) + f(a)$$

3. If y = distance (ft) and x = time (sec), then is $f'(x) =$ velocity (ft/sec).

Original	***Derivative***
Horiz. Tangent	Zero (f'(x) = 0)
Increasing	Positive
Decreasing	Negative
Vertical Tangent	Undefined

10.2 Calculus on Parametric Curves

Given $x = x(t), y = y(t)$



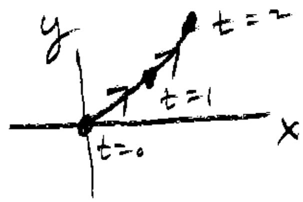
1. x = distance, y = distance, t = time
2. $\dfrac{dx}{dt} = x'(t) =$ horiz. velocity
3. $\dfrac{dy}{dt} = y'(t) =$ vert. velocity

Original	***Derivatives***
Horiz. Tangent	y'(t) = 0
Moving Upward	y'(t) positive
Moving Down	y'(t) negative
Vert. Tangent	x'(t) = 0
Moving Right	x'(t) positive
Moving Left	x'(t) negative

Example: $x(t) = \frac{1}{2}t,\ y(t) = t^2 + 10t$

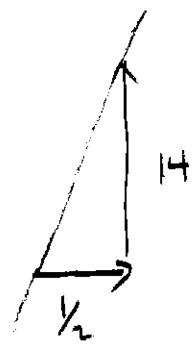
1. Plot the (x, y) points corresponding to $t = 0, t = 1$, and $t = 2$.
2. Find the formulas for $\frac{dx}{dt}$ and $\frac{dy}{dt}$.
3. Compute $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at $t = 2$.
4. Eliminate the parameter to find the equation for the “curve on which the motion is occurring” in the form $y = f(x)$.
5. Give the equation of the tangent line when *t* = 2.

t	0	1	2
x	0	½	1
y	0	11	24



$\frac{dx}{dt} = \frac{1}{2}$, $\frac{dy}{dt} = 2t + 10$

$\left.\frac{dx}{dt}\right|_{t=2} = \frac{1}{2}$, $\left.\frac{dy}{dt}\right|_{t=2} = 14$



$x = \frac{1}{2}t \Rightarrow t = 2x$

$y = t^2 + 10t \Rightarrow y = (2x)^2 + 10(2x)$

$\boxed{y = 4x^2 + 20x}$

$\frac{dy}{dx} = 8x + 20$ slope

$\left.\frac{dy}{dx}\right|_{(x,y)=(1,24)} = 8(1) + 20 = 28$

$\boxed{y = 28(x-1) + 24}$

NOTE: $\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{14}{\frac{1}{2}} = 28 \leftarrow$ INTERESTING

Big fact: "Proof" of fact that $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Assume $x = x(t), y = y(t)$ describes motion along the curve $y = f(x)$.

Then at all times $y(t) = f(x(t))$.

By the chain rule: $y'(t) = f'\big(x(t)\big)x'(t)$, that is, $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

Therefore, $\frac{y'(t)}{x'(t)} = f'\big(x(t)\big)$, which is the same as $\frac{dy/dt}{dx/dt} = \frac{dy}{dx}$

Example (HW10.2 #3):

$x = t - t^{-1}, y = 9 + t^2$

Find the equation for the tangent line when *t = 1*.

$x(1) = 1 - \frac{1}{1} = 0$

$y(1) = 9 + 1^2 = 10$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{1 + t^{-2}}$

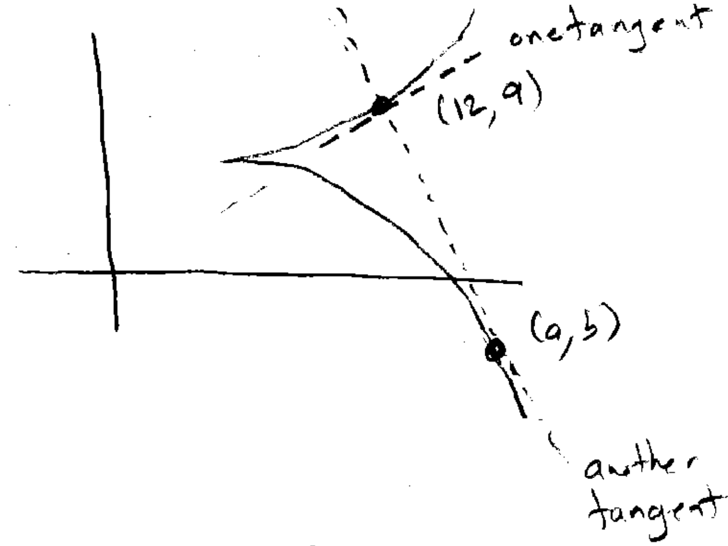
$\left.\frac{dy}{dx}\right|_{t=1} = \frac{2}{1+1} = 1$

$y = 1(x - 0) + 10$

$\boxed{y = x + 10}$

HW10.2 #7 Hint:

$x = 9t^2 + 3, y = 6t^3 + 3$ $\Longleftrightarrow$



There are two tangent lines to this curve that **also** pass through (12,9).

Find these two tangent line.

I (a,b) IS ON CURVE $\Rightarrow$?

II DESIRED SLOPE $= \dfrac{b-9}{a-12}$

III TANGENT SLOPE $= \dfrac{dy}{dx} = ?$

Hint: (12,9) is on the curve (when?). So you can find one tangent line quickly. But there is another point is unknown (a,b). You will need to solve for (a,b) (like we have done in other problems)

You do!

Old Final Question

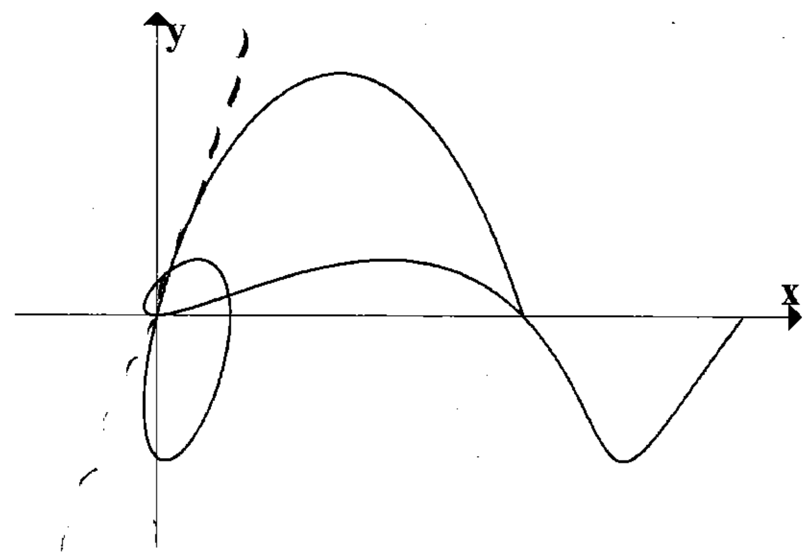
A particle is moving in the xy-plane according to the equations:

$$x(t) = \cos(\pi t) + t^2 \qquad y(t) = 2(t-1)\sin((t+1)\pi)$$

Find the equation for the tangent line when t = -1.

$x(-1) = \cos(-\pi) + (-1)^2 = -1 + 1 = 0$

$y(-1) = 2(-1-1)\sin((-1+1)\pi) = 0$



$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{2(t-1)\cos((t+1)\pi)\pi + 2\sin((t+1)\pi)}{-\pi\sin(\pi t) + 2t}$$

$$\left.\frac{dy}{dx}\right|_{t=-1} = \frac{2(-1-1)\cos((0)\pi) + 2\sin((0)\pi)}{-\pi\sin(-\pi) - 2}$$

$$= \frac{-4+0}{-2} = 2$$

$$y = 2(x-0) + 0 \Rightarrow \boxed{y = 2x}$$

Speed: For a parametric equation, it is natural to ask what the “speedometer” speed is for the moving object (the speed in the direction it is moving).

“Proof sketch” that $\text{speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

Assume $x = x(t), y = y(t)$ describes motion along a curve.

“average speed from *t* to *t+h*” $= \dfrac{\text{change in distance}}{\text{change in time}}$

$$\approx \frac{\sqrt{\left(x(t+h) - x(t)\right)^2 + \left(y(t+h) - y(t)\right)^2}}{h}$$

$$= \sqrt{\left(\frac{x(t+h) - x(t)}{h}\right)^2 + \left(\frac{y(t+h) - y(t)}{h}\right)^2}$$

“instantaneous speed at *t*” is the limit of the above expressions as $h \to 0$

Example: $x(t) = \frac{1}{2}t,\ y(t) = t^2 + 10t$

1. What is the formula for speed?
2. What is the speed at t = 2?

$\boxed{1}$ $\sqrt{\left(\frac{1}{2}\right)^2 + (2t+10)^2} = \text{speed}$

$\boxed{2}$ at $t = 2$

$\sqrt{\left(\frac{1}{2}\right)^2 + (2(2)+10)^2}$

$\sqrt{1/4 + 14^2} = \sqrt{196.25} \approx 14.0089 \ \frac{\text{ft}}{\text{sec}}$

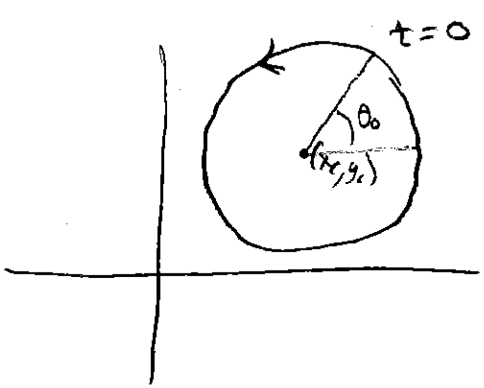
Special parametric equations:

1. Uniform Circular Motion:

$$(x_c, y_c) = \text{center of circle}$$
$$r = \text{radius}, \theta_0 = \text{initial angle}$$
$$\omega = \text{angular speed } \left(\frac{\text{rad}}{\text{time}}\right)$$
$$\boldsymbol{x = x_c + r\cos(\,\theta_0 + \omega t)}$$
$$\boldsymbol{y = y_c + r\sin(\,\theta_0 + \omega t)}$$

Note the fundamental facts about circular motion (*only* true in radians):

$$\text{linear speed} = v = \omega r,$$
$$\text{arc length} = s = r\theta$$



Ex)

If $\omega = 2\,\frac{\text{rad}}{\text{sec}}$ } angular speed

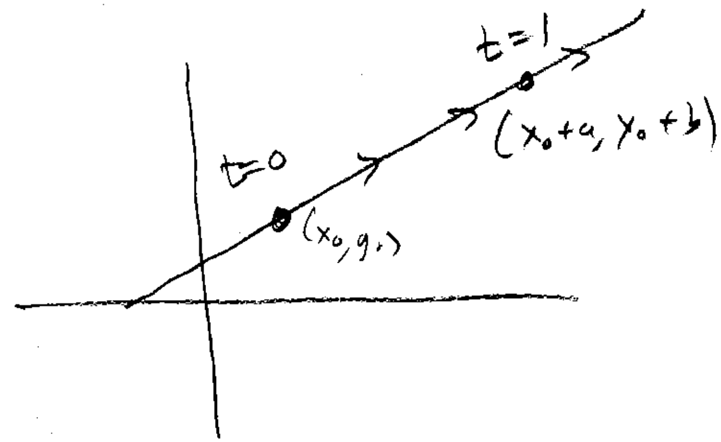
and $r = 5$ inches

then

$$\underbrace{v = \omega r = 10\ \frac{\text{inches}}{\text{sec}}}_{\text{linear speed}}$$

2. Uniform Linear Motion:

$$(x_0, y_0) = \text{initial location}$$
$$a = \text{horiz. velocity}$$
$$b = \text{vert. velocity}$$
$$\boldsymbol{x = x_0 + at}$$
$$\boldsymbol{y = y_0 + bt}$$



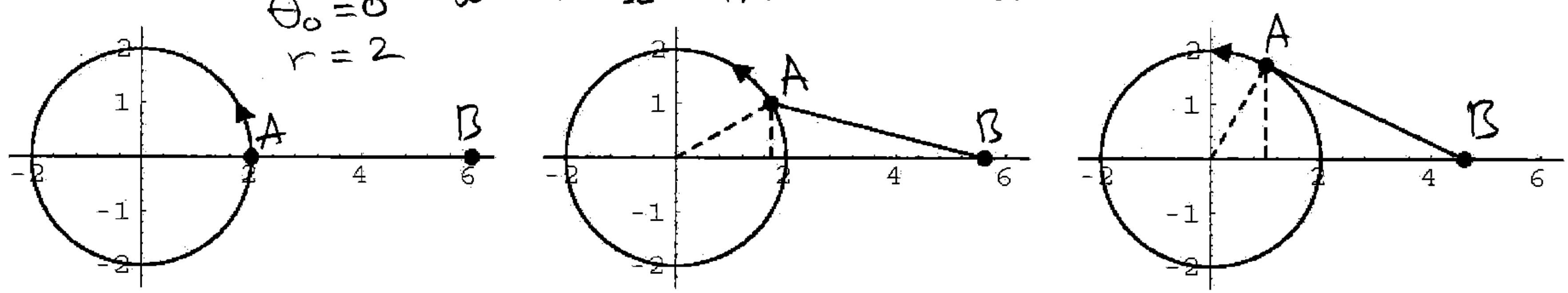
Directly from homework:

A 4-centimeter rod is attached at one end A to a point on a wheel of radius 2 cm. The other end B is free to move back and forth along a horizontal bar that goes through the center of the wheel. At time *t=0* the rod is situated as in the diagram at the left below. The wheel rotates counterclockwise at 3.5 rev/sec. Thus, when *t=1/21* sec, the rod is situated as in the diagram at the right below.

$\omega = 3.5 \frac{\text{REV}}{\text{SEC}} \frac{2\pi \text{ RAD}}{1 \text{ REV}} = 7\pi \frac{\text{RAD}}{\text{SEC}}$

$\theta_0 = 0$

$r = 2$



Find parametric equation for the point A and the point B.

$\boxed{A}$: $x = 2\cos(7\pi t)$

$y = 2\sin(7\pi t)$

$\boxed{B}$ $x = ?$

$y = 0$

THE DIST. FROM A to B IS <u>ALWAYS</u> 4

$$\Rightarrow \sqrt{(x - 2\cos(7\pi t))^2 + (0 - 2\sin(7\pi t))^2} \overset{?}{=} 4$$

Solve for x!